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Translation invariant states on twisted algebras on a lattice

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Abstract
We construct an algebra with twisted commutation relations and equip it with the shift. For appropriate irregularity of the non-local commutation relations, we prove that the tracial state is the only translation-invariant state.

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1. Introduction

Whereas the dynamics of quantum systems with finitely many degrees of freedom is fully controllable via the spectrum of the unitary operator that implements it, this is no longer true with infinitely many degrees of freedom. While these systems can always be described by operators acting on suitable Hilbert spaces, it may however happen, typically in the thermodynamic limit and in the presence of interactions, that not all of these operators represent observable quantities.

In such cases, one prefers to focus upon equilibrium (time-invariant) states characterized by a few parameters as temperature and chemical potential. There thus arises the question whether such states cover essentially all of the system time-invariant states or, if there are other time-invariant states, whether and why these latter should not be as physically important.

Consider, for instance, the case of a quasi-free time evolution of non-interacting particles determined by a single-particle Hamiltonian h; the time evolution of the infinite system is an automorphism α_h. If k is any other single-particle Hamiltonian commuting with h ($[h, k] = 0$), then the automorphisms α_h and α_k share the same time-invariant states $\omega = \omega \circ \alpha_h = \omega \circ \alpha_k$, which therefore need infinitely many parameters to be characterized.

There is, however, the conviction that interactions reduce the possible equilibrium states via a mechanism that should be rather general in its yet unknown relevant features. Unfortunately, there is so far scarcely any control on the asymptotic properties of the dynamics of realistic physical systems. Some clues are the following ones: space translations teach us that a too strong asymptotic (with respect to repeated applications α_h^n of the automorphism) independence, or asymptotic commutativity (Abelianess), cannot be the essential feature. For instance, in the case of translation-invariant time evolutions, norm asymptotic Abelianess, namely

$$\lim_{n\to\infty} \left\| \left[A, \alpha_h^n(B)\right] \right\| = 0 \qquad \forall A, B \in \mathcal{A},$$

would yield a trivial theory. On the other hand, Galilei-invariant time evolutions which posses infinitely many equilibrium states are essentially weakly asymptotically Abelian with respect to a given equilibrium state ω [1], that is

$$\lim_{n\to\infty} \omega\left(C^\dagger\left[A, \alpha_h^n(B)\right]D\right) = 0 \qquad \forall A, B, C, D \in \mathcal{A}.$$

It is thus of interest to examine toy models for which the dynamics can be proved to be weakly asymptotically Abelian and which, none the less, have a relatively small set of invariant states.

A particular instance of the equilibrium state is the so-called tracial, or totally mixed, state ω on a (C*) algebra $\mathcal{A}$, whose two-point correlation functions are such that $\omega(AB) = \omega(BA)$ for all $A, B \in \mathcal{A}$. There have already been constructed non-commutative systems where the tracial state is the only invariant state for certain automorphisms; examples are the Price–Powers shifts [2–5] and the irrational rotation algebras [5, 6]. In both cases, what forbids the existence of invariant states different from the tracial one is that operators in the course of time anti-commute infinitely often and sufficiently irregularly with one another. Such a lack of asymptotic commutativity is indeed expected in real interacting quantum systems.

In the following, we give another example of such quantum dynamical systems inspired very much by the Price–Powers shifts. However, the way how asymptotic commutativity is violated is less restrictive. For the Price–Powers shift, the algebra $\mathcal{A}$ is created from self-adjoint operators $e_k = e_k^*$, $e_k^2 = \mathbb{1}$, $k \in \mathbb{N}$, that commute or anticommute,

$$e_k e_p = (-1)^{g(|p-k|)} e_p e_k,$$

as prescribed by a so-called *bitstream*, namely a two-valued function on the integers, $g : \mathbb{N} \mapsto \{0, 1\}$, $g(0) = 0$.

We shall instead consider Weyl operators in place of the e_k and organize them in such a way that shifted Weyl operators remain in Weyl-like relation with one another in a so-called *complementary* manner [7] so that any two of them create a full matrix algebra $M_{d\times d}$. We shall show that, under the hypothesis of sufficiently irregular complementary Weyl-like relations, the tracial state is the only translationally invariant state as for the Powers–Price shifts.

The paper is organized as follows. In section 2, we define the algebra and the complementary relations, together with some representations either as an AF-algebra (in some cases a UHF algebra) or as a quantum-spin chain by means of generalized Jordan–Wigner transformations. In section 3, we show that under the assumption of sufficiently random commutation relations only the tracial state can be translation invariant.

2. The algebra and its automorphism

We start with infinitely many finite-dimensional algebras $\mathcal{A}_m$, $m \in \mathbb{N}$, all isomorphic to $d \times d$ matrix algebras, created by the operators $W_{\vec{k}}^{(m)}$, $\vec{k} \in \mathbb{Z}_d^2 := \{(k_1, k_2),\ k_i = 0, 1, \ldots, d-1\}$, that satisfy the commutation relations of a discrete Weyl group

$$W_{\vec{k}_1}^{(m)} W_{\vec{k}_2}^{(m)} = \mathrm{e}^{\frac{2\pi i \sigma(\vec{k}_1, \vec{k}_2)}{d}} W_{\vec{k}_2}^{(m)} W_{\vec{k}_1}^{(m)}, \tag{1}$$

with a symplectic form $\sigma(\vec{k}_1, \vec{k}_2) := k_{11}k_{22} - k_{12}k_{21}$.

Note that $W_{\vec{0}}^{(m)} = \mathbb{1}$ and $W_{-\vec{k}}^{(m)} = \left(W_{\vec{k}}^{(m)}\right)^{-1}$.

The relations between Weyl operators with different upper indices are *twisted* by means of a sequence of 2×2 matrices A_n, $n \in \mathbb{Z}$, with entries in $\{0, 1, \ldots, d-1\}$; explicitly,

$$W^{(p)}_{\vec{k}_p} W^{(q)}_{\vec{k}_q} = \mathrm{e}^{2\pi \mathrm{i} u_{\vec{k}_p \vec{k}_q}(q-p)} W^{(q)}_{\vec{k}_q} W^{(p)}_{\vec{k}_p}, \tag{2}$$

$$u_{\vec{k}_p \vec{k}_q}(q-p) := \frac{1}{d}\sigma(\vec{k}_p, A_{q-p}\vec{k}_q). \tag{3}$$

Setting $A_0 = \mathbb{1}$, the single-site relations (1) are a particular instance of (2).

The finite products define elements of an infinite discrete group. We denote them as

$$W_I := W^{(1)}_{\vec{k}_1} W^{(2)}_{\vec{k}_2} \cdots W^{(\ell)}_{\vec{k}_\ell}, \tag{4}$$

where I denotes a sequence of vectors $\{\vec{k}_m\}_{m\in\mathbb{Z}}$ with only finitely many components, $(\vec{k}_1, \vec{k}_2, \ldots, \vec{k}_\ell)$, possibly different from the vector $\vec{0}$. We define a star operation as $\left(W^{(m)}_{\vec{k}}\right)^* = \left(W^{(m)}_{\vec{k}}\right)^{-1}$ and $(UV)^* = V^* U^*$ as usual. So the Weyl operators and their products are unitary elements of the C^* algebra generated by the finite products W_I, which are assumed to be linearly independent. We shall denote this algebra by $\mathcal{A}$.

We shall further equip $\mathcal{A}$ with an automorphism $\alpha : \mathcal{A} \mapsto \mathcal{A}$ such that $\alpha^n\left(W^{(0)}_{\vec{k}}\right) = W^{(n)}_{\vec{k}}$. Then, generic algebraic relations read

$$W_I \alpha^n(W_J) = \left(\prod_{a=1}^{n_I} W^{(a)}_{\vec{k}_a}\right)\left(\prod_{b=1}^{n_J} W^{(b+n)}_{\vec{k}_b}\right) = \mathrm{e}^{2\pi \mathrm{i} u_n(I;J)} \alpha^n(W_J) W_I, \tag{5}$$

$$u_n(I;J) := \frac{1}{d}\sum_{a=1}^{n_I}\sum_{b=1}^{n_J} \sigma(\vec{k}_a, A_{b+n-a}\vec{k}_b) = \sum_{a=1}^{n_I}\sum_{b=1}^{n_J} u_{\vec{k}_a \vec{k}_b}(b+n-a). \tag{6}$$

Remarks 1

(1) Every sequence of 2×2 matrices $\{A_n\}_{n\in\mathbb{N}}$ generates its own algebra $\mathcal{A}$; however, for special sequences $\{A_n\}_{n\in\mathbb{N}}$ (as will be shown in the following section), the corresponding algebras built by $W^{(m)}_{\vec{k}}$, $m = 1, \ldots, l$ will, for all l, be isomorphic to the tensor product over l local sites, and therefore the total algebra can be considered as the same $\mathcal{A}$ equipped with the usual shift and a different automorphism α with finite speed.

(2) We will refer to the Weyl operators $W^{(m)}_{\vec{k}}$ in (1) as to the letters of the algebra $\mathcal{A}$ and to the products W_I as in (4) as to the words of $\mathcal{A}$. The norm of every word equals 1 and $(W_I)^d = \mathbb{1}$, as the eigenvalues of all W_I are the pure phases $\mathrm{e}^{\frac{2\pi \mathrm{i}\ell}{d}}$, $0 \leqslant \ell \leqslant d-1$. Linear combinations of words will be referred to as sentences, as in the case of the Price-Powers-shift. A state $\omega : \mathcal{A} \mapsto \mathcal{C}$ over the algebra amounts to a positive, normalized functional over $\mathcal{A}$: it is thus fixed by giving its values on all words.

We proceed in studying the algebras, showing the existence of a non-trivial representation. This is not a trivial problem, see [8]. Consider the set $\mathcal{I}$ of multi-indices I and define the composition law of addition: $\mathcal{I} \times \mathcal{I} \mapsto \mathcal{I}$: if $I = \left\{\vec{k}^I_m\right\}_{m\in\mathbb{Z}}$ and $J = \left\{\vec{k}^J_m\right\}_{m\in\mathbb{Z}}$, then

$$(I, J) \mapsto I + J = \left\{\vec{k}^I_m + \vec{k}^J_m\right\}_{m\in\mathbb{Z}},$$

where the sum of vectors $\vec{k}^I_m + \vec{k}^J_m$ is understood modulo d. Then, the family of operators $\{\widetilde{W}_I\}_{I\in\mathcal{I}}$ satisfying the multiplication law $\widetilde{W}_I \widetilde{W}_J = \widetilde{W}_{I+J}$ forms an Abelian group G on which the shift defines an automorphism with an associated shift-invariant measure δ such that $\delta(\widetilde{W}_I) = 0$ unless $I = I_0 := \{\vec{0}\}_{m\in\mathbb{Z}}$. Therefore, one can consider the Hilbert space $\ell_2(G)$

spanned by the orthonormal elements $\widetilde{W}_I$ and represent the Weyl operators W_I introduced before by

$$\Pi(W_I)\widetilde{W}_J = \underbrace{\mathrm{e}^{\mathrm{i}\pi u_0(I,J)}}_{\omega(I;J)}\widetilde{W}_{I+J}. \tag{7}$$

$\omega(I;J)$ is a co-cycle, namely

$$\omega(I_1, I_2 + I_3)\omega(I_2, I_3) = \omega(I_1 + I_2, I_3)\omega(I_1, I_3).$$

In this way, $\mathcal{A}$ is represented as a sub-algebra $\Pi(\mathcal{A}) \subseteq \mathcal{B}(\ell_2(G))$ (it is known as the regular representation) of the bounded operators on $\ell_2(G)$ and thus all considerations in [9] are therefore applicable to $\mathcal{A}$. For instance, $\mathcal{A}$ has a trivial center if to any word W_I there exists another word W_J such that W_I and W_J do not commute. In this case the trace on the algebra

$$tr(W_I) = 0 \qquad \forall I \neq \emptyset, \qquad tr(\mathbb{1}) = 1 \tag{8}$$

is unique, and it is implemented in the regular representation by

$$tr(W_I) = \langle \widetilde{W}_{I_0} | \Pi(W_I) | \widetilde{W}_{I_0} \rangle.$$

Evidently, the trace is invariant under the shift automorphism $\alpha : \mathcal{A} \mapsto \mathcal{A}$

Like in [9], the main interest is in finding conditions on the co-cycle (7) such that no other invariant states exist other than the tracial state. In [9] the main tool was the high degree of anticommutativity, a generalization of the fact, that translation invariant states over Fermi systems have to be even. Though this criterion is sufficient only if $d = 2$, non-commutativity as embodied in (2) will nevertheless turn out to be just as powerful in restricting the class of invariant states. We will indeed give other arguments to enlarge the class of automorphisms that allow only the tracial state as an invariant state. Though not optimal, the result indicates that dynamical delocalization as described in the following section is an effect which is worth studying in more detail and in more realistic thermodynamic systems.

2.1. Spin-chain representation

We have already given a representation of the algebra over $\mathcal{B}(l_2(G))$. However, in order to bring it in closer contact with physical models, we seek connections with spin chains and their automorphisms.

This demands different representations. To demonstrate the differences we make a short detour, considering finite algebras defined on a finite ring of N lattice points (= upper indices) with a cyclic shift, instead of an infinite set of points with a non-recurrent shift. We get different dimensions of the Hilbert spaces. In the above-mentioned representation there are $(d^2)^N$ basis vectors. For a spin chain one would expect only d^N as necessary. But this, as will turn out, is possible for a restricted set of defining sequences only. In section 2.2, we give then a representation with a double-spin chain, possible for any defining sequence, employing again $(d^2)^N$ basis vectors for the finite algebra on a ring.

More precisely, we shall try to represent the Weyl operators $W^{(m)}_{\vec{k}}$, $m = 0, \ldots, N$, as elements of the full matrix algebra $\bigotimes_{n=0}^{N}(M_{d\times d})_n$. We proceed step by step: let us define the Weyl operator at site $0 \leqslant j \leqslant N$ to be

$$W^{(j)}_{\vec{k}} = W_{A_{0,j}\vec{k}} \otimes W_{A_{1,j}\vec{k}} \otimes W_{A_{j,j}\vec{k}} \otimes \mathbb{1}_{j+1} \ldots \otimes \mathbb{1}_N. \tag{9}$$

The unknowns in the construction are the 2×2 matrices with integer entries from $\{0, 1, \ldots, d-1\}$ that we have to adjust in order to fulfill the commutation relations (2). Therefore, from (1), one gets the condition

$$\sum_{\ell=0}^{j} \sigma(A_{\ell,j}\vec{k}, A_{\ell,j}\vec{m}) = \sigma(\vec{k}, \vec{m}), \tag{10}$$

for all $\vec{k}$, $\vec{m}$ and $0 \leqslant j \leqslant N$ which is equivalent to $\sum_{\ell=0}^{j} \mathrm{Det}(A_{\ell,n}) = 1$. If d is prime, then all Weyl operators are unitarily isomorphic so that the algebra created by them is $M_{d\times d}$; therefore, in the rest of this section, d will be assumed to be a prime number.

We have to control whether this ansatz can really be satisfied and how far the matrices $A_{\ell,k}$ are determined by the matrices A_n in (3). It turns out that $A_{0,j} = A_j$ while $A_{00} = \mathbb{1}$. The other matrices $A_{\ell,k}$ have to be calculated recursively from (2) and (3). More precisely to evaluate

$$\sigma(A_{0,1}\vec{k}, A_{0,2}\vec{l}) + \sigma(A_{1,1}\vec{k}, A_{1,2}\vec{l}) = \sigma(\vec{k}, A_{0,1}\vec{l}), \tag{11}$$

we define the linear map $A \mapsto \widehat{A}$,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \rightarrow \widehat{A} := \begin{pmatrix} a_{22} & -a_{12} \\ a_{21} & a_{11} \end{pmatrix}.$$

With this map, it follows that $A_{1,2} = \hat{A}_{1,1}^{-1}(A_{0,1} - \hat{A}_{0,1}A_{0,2})$. This fixes $A_{1,2}$ if we take into account that the freedom in $A_{1,1}$ is reduced to an isomorphism inside of the local algebra. However, we can only be sure that there exists a solution $A_{1,2}$ if $A_{1,1}$ is invertible which surely holds if $\mathrm{Det}(A_{1,1}) \neq 0$. Similarly

$$A_{n-k,n} = \hat{A}_{n-k,n-k}^{-1}\left(A_{0,n-k} - \sum_{l=0}^{n-k-1} \hat{A}_{l,n-k}A_{l,n}\right).$$

The equations are uniquely solvable (up to trivial local isomorphisms) under the constraint

$$\mathrm{Det}(A_{0,j}) + \cdots + \mathrm{Det}(A_{j-1,j}) \neq 1 \quad \forall j. \tag{12}$$

This implies in addition that $\mathrm{Det}(A_{j,j}) \neq 0$; as a consequence the algebra generated by the Weyl operators (9) is isomorphic to the full matrix algebra.

Only the sequence of matrices A_n is at our disposal, whereas the matrices $A_{\ell,k}$ are linearly depending on them; however, there remains enough freedom to find sequences that meet condition (12). A special example corresponds to choosing $A_{0,j} = \delta_{1,j}\mathbb{1}$; this in turn corresponds to the usual shift on the lattice algebra. More generically, one may choose $A_{0,j}$ to belong to a left ideal with determinant 0 for all $j \neq 0$; it then follows that, for all $k > 0$, $A_{n-k,n}$ also belongs to this ideal and therefore $\mathrm{Det}(A_{n,n}) = 1 \quad \forall n$.

As a consequence, provided $\mathrm{Det}(A_{n,n}) \neq 0 \quad \forall n$, we can consider the algebra to be the same algebra i.e. the spin chain $M_{d\times d}^{\otimes\infty}$, but equipped with different automorphisms corresponding to the different sequences $A_{0,n}$. Note that in the spin-chain representation, the algebra is fairly simple, while the automorphism (which we will again denote by α and which corresponds to the shift in the regular representation) is complicated.

Some discussions on the spin chain representations and their importance for physics follows in the conclusion.

2.2. *The generalized Jordan–Wigner transformation*

We have already mentioned the representation of every algebra $\mathcal{A}$ as a C^* algebra $\Pi(\mathcal{A}) \subseteq \mathcal{B}(\ell_2(G))$. We can give another representation that can be considered as a generalization of the Jordan–Wigner transformations that relates the spin lattice to the fermions on a lattice. We represent $\mathcal{A}$ as a subalgebra of the doubled-spin chain $\bigotimes_{m=-\infty}^{\infty}(M_{d\times d} \otimes M_{d\times d})_m = \bigotimes_{n=-\infty}^{\infty}(M_{d\times d})_n$, where $W_{\vec{k}}^{(0)}$ with $\vec{k} = (k_1, k_2)$ is identified with the infinite tensor product

$$\left(\bigotimes_{n=1}^{+\infty} \left(W_{0,b_n}\right)_{-2n} \otimes \left(W_{0,a_n}\right)_{-2n+1}\right) \otimes \left(W_{k_1,0}\right)_0 \otimes \left(W_{k_2,k_1}\right)_1 \bigotimes_{n=2}^{+\infty}(\mathbb{1})_n, \tag{13}$$

where the Weyl operator $W_{k_1,0}$ is at site $n = 0$, while W_{k_2,k_1} is at site $n = 1$, whereas the operators W_{0,b_n} are located at sites $-2n$ and W_{0,a_n} at sites $-2n+1$. Moreover, the components a_n and b_n are determined by $\vec{k}$ and the commutation relations via

$$A_n \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} b_n \\ a_n \end{pmatrix}.$$

Furthermore, the action of the shift automorphism α is now represented as a two-step translation along the lattice.

Note that since the contributions from the infinite tails commute with each other by construction, finite tensor products of the form $W^{(0)}_{k_1,k_2} \cdots W^{(N)}_{l_1,l_2}$ may be effectively represented as elements of the matrix algebra $\bigotimes_{n=0}^{2N} (M_{d\times d})_n$. It thus follows that the commutant consists of operators of the form

$$\bigotimes_{n=2j-1}^{-\infty} (\mathbb{1})_n \otimes \left(W_{\ell_1,\ell_2}\right)_{2j} \otimes \left(W_{\ell_2,0}\right)_{2j+1} \otimes \left(\bigotimes_{k=j+1} \left(W_{0,b_{k-j}(\ell)}\right)_{2k} \otimes \left(W_{0,a_{k-j}(\ell)}\right)_{2k+1} \right),$$

plus operators belonging to the center (the center becomes trivial if (12) holds; however, this condition is not needed here.)

We can consider the operators to act on the vector $|\Omega >= \otimes_{-\infty<k<\infty}|0>$ where at each lattice point $W_{0,k}|0>=|0>$. Representing $\widetilde{W}^0_{k_1,k_2}$ by

$$\left(\bigotimes_{n=1}^{+\infty} \mathbb{1} \otimes \mathbb{1} \right) \otimes \left(W_{k_1,0}\right)_0 \otimes \left(W_{k_2,0}\right)_1 \bigotimes_{n=2}^{+\infty} (\mathbb{1})_n$$

and letting it acting on $|\Omega >= \left|\widetilde{W}_{I_0}\right\rangle$, we reproduce $l^2(G)$ and therefore the regular representation.

Example 1. As a concrete illustration of the algebraic construction of above, let us consider the Price–Powers shift. As mentioned in the introduction, this corresponds to the shift on the C^* algebra generated by products of self-adjoint operators e_j, $j \in \mathbb{Z}$ such that

$$e_k e_p = (-1)^{g(|p-k|)} e_p e_k.$$

The algebraic relations can be implemented by means of the Pauli matrices as follows:

$$e_k = \bigotimes_{n=1}^{+\infty} \left(\sigma_3^{g(k)}\right)_{k-n} \otimes (\sigma_1)_k \bigotimes_{n=k+1}^{+\infty} (\mathbb{1})_n,$$

where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$.

Instead, within the Weyl framework, one has $d = 2$ and a generating sequence of matrices which are either $A_n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ corresponding to $g(n) = 0$ or $A_n = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ corresponding to $g(n) = 1$. Then, setting $e_0 = W^{(0)}_{1,0}$, one observes that operators at odd places have the form $W_{0,k}$ and therefore all commute. Consequently, they can be removed from the tensor product (13), so that, finally one can represent

$$e_k = \bigotimes_{n=1}^{+\infty} \left(W_{0,b_n}\right)_{k-n} \otimes \left(W_{1,0}\right)_k \bigotimes_{n=k+1}^{+\infty} (\mathbb{1})_n.$$

This corresponds to choosing $W_{0,0} = 1$, $W_{0,1} = \sigma_3$, $W_{1,0} = \sigma_1$, $W_{1,1} = -\mathrm{i}\sigma_2$.

2.3. Preliminary remarks on invariant states

We now turn to the problem of finding the invariant states under the two-step shift. If we are only interested in the local effects of such automorphism, we can take the periodic shift in $\bigotimes_{n=0}^{2N}(M_{d\times d})$, which is unitarily implemented.

The algebra created by finitely many Weyl operators is imbedded in $\bigotimes_{n=0}^{2N}(M_{d\times d})$, so that we can conclude that the automorphism $\alpha : \mathcal{A} \mapsto \mathcal{A}$ is unitarily implemented. Therefore, we can construct states on the local algebra defined by density operators that commute with the unitary that implements the periodic shift. However, in general, these density operators will not have a limit when $N \to \infty$.

Another possibility of constructing α-invariant states is to start from vectors in the infinite tensor product $\bigotimes_{n=-\infty}^{\infty} |\psi_n\rangle$ that are invariant under the shift. The simple choice where all $|\psi_n\rangle$ are identical to an eigenvector $|\phi\rangle$ of $W_{(0,1)}$ gives, independently of the sequence $\{A_n\}_{n\in\mathbb{N}}$, already the tracial state; indeed, since for every W_I at some position $\langle\phi|W_{k,l}|\phi\rangle = 0$. Therefore, the representation is isomorphic to the regular representation.

If we choose some other vector and assume that $A_n \neq \mathbb{1}$ for infinitely many n, again we obtain the tracial state; in fact, either $|\langle\psi|W_{0,b_n}|\psi\rangle\langle\psi|W_{0,a_n}|\psi\rangle| < 1$, infinitely often so that

$$\prod_{n=-\infty}^{+\infty} |\langle\psi|W_{0,b_n}|\psi\rangle\langle\psi|W_{0,a_n}|\psi\rangle| = 0.$$

If for some $N > 0$ $A_n = 0$ for all $n > N$, we can choose a vector ϕ over $\bigotimes_{n=1}^{2N} M_{d\times d}^{(n)}$ that is appropriately entangled over the lattice points to guarantee that $\langle\phi|W_{k_1,0)}|\phi\rangle \neq 0$. By averaging this vector over the period N, one gets an expectation value still $\neq 0$. However, in general, one expects that it decreases with N. Therefore, if for every N we can find $n > N$ such that $A_n \neq 0$ in order to obtain another invariant state in the limit $N \to \infty$, it is necessary to have correlations between infinitely many lattice points; this can hardly be satisfied because of monogamy of entanglement. Though we are unable to exclude that other invariant states might be constructed, our considerations already indicate that the tracial state will turn out to be the only invariant state under appropriate conditions on the defining sequence of matrices A_n.

3. Invariant states

Given $(\mathcal{A}, \alpha)$, let ω be an invariant state such that $\omega \circ \alpha = \omega$ and consider the corresponding GNS representation π_ω of $\mathcal{A}$ as a C^* algebra of bounded operators on the GNS Hilbert space $\mathcal{H}_\omega$ with a cyclic vector $|\Omega\rangle$. Namely, $\omega(A) = \langle\Omega|\pi_\omega(A)|\Omega\rangle$; further, the shift automorphism α is implemented by a unitary U_ω such that $U_\omega|\Omega\rangle = |\Omega\rangle$. The following simple lemmas hold.

Lemma 1. *Let P_0^ω denote the projection onto the U_ω-invariant subspace of $\mathcal{H}_\omega$. If*

$$\langle\Omega|\pi_\omega(W_I^*)P_0^\omega\pi_\omega(W_I)|\Omega\rangle = 0 \tag{14}$$

is true for all W_I, then ω is tracial that is $\omega(XY) = \omega(YX)$ for all $X, Y \in \mathcal{A}$.

Proof. Since $P_0^\omega \geqslant |\Omega\rangle\langle\Omega|$, the assumption implies

$$|\omega(W_I)|^2 = \langle\Omega|\pi_\omega(W_I^*)|\Omega\rangle\langle\Omega|\pi_\omega(W_I)|\Omega\rangle \leqslant \langle\Omega|\pi_\omega(W_I^*)P_0^\omega\pi_\omega(W_I)|\Omega\rangle = 0,$$

whence, by Cauchy–Schwartz, $\omega(W_IW_J) \neq 0$ only if $W_IW_J = \mathbb{1}$. Thus, $\omega(W_IW_J) = \omega(W_JW_I)$ for all $W_{I,J}$, whence $\omega(XY) = \omega(YX)$ for $\mathcal{A}$ is generated by linear combinations of W_Is. □

Lemma 2. *If ω is α-invariant, then, setting $u_n(I) := u_n(I; I)$ in (6),*

$$\langle\Omega|\pi_\omega(W_I^*)P_0^\omega\pi_\omega(W_I)|\Omega\rangle = \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\mathrm{e}^{2\pi\mathrm{i}u_n(I)}\langle\Omega|\pi_\omega(W_I)U_\omega^{-n}\pi_\omega(W_I^*)|\Omega\rangle. \tag{15}$$

Proof. The mean ergodic theorem of von Neumann ([10]) and (6) imply

$$\begin{aligned}\langle\Omega|\pi_\omega(W_I^*)P_0^\omega\pi_\omega(W_I)|\Omega\rangle &= \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\omega(W_I^*\alpha^n(W_I))\\ &= \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\mathrm{e}^{2\pi\mathrm{i}u_n(I)}\omega(W_I\alpha^{-n}(W_I^*)).\end{aligned}$$

□

Remarks 2

(1) If $W_I = \mathbb{1}$, then $\langle\Omega|\pi_\omega(W_I^*)P_0^\omega\pi_\omega(W_I)|\Omega\rangle = 1$, $u_n(I) = 0$ for all $n \in \mathbb{N}$ and (15) is trivially satisfied.

(2) Using the spectral decomposition $U_\omega = \int_{\mathrm{Sp}(U_\omega)} \mathrm{d}P_\omega^\lambda\, \mathrm{e}^{2\pi\,\mathrm{i}\lambda}$, (15) reads

$$\langle\Omega|\pi_\omega(W_I^*)P_0^\omega\pi_\omega(W_I)|\Omega\rangle = \lim_{N\to\infty}\int_{\mathrm{Sp}(U_\omega)}\mathrm{d}\mu_I(\lambda)\frac{1}{N}\sum_{n=0}^{N-1}\mathrm{e}^{2\pi\mathrm{i}(u_n(I)-n\lambda)}, \tag{16}$$

where $\mathrm{d}\mu_I(\lambda) := \mathrm{d}(\langle\Omega|\pi_\omega(W_I)P_\omega^\lambda\pi_\omega(W_I^*)|\Omega\rangle$.

We now concentrate on the sequences $u(I) := \{u_n(I)\}_{n\in\mathbb{N}}$ and $u_n^\lambda(I) := \{u_n(I) - \lambda n\}_{n\in\mathbb{N}}$ and study their spectrum [10]. In order to properly introduce this notion, consider a sequence $v = \{v_n\}_{n\in\mathbb{N}}$ taking its values $v_n \in \mathbb{C}$ in a compact subset of the complex numbers. For all $k \in \mathbb{N}$, the partial sums

$$S_N(k) := \frac{1}{N}\sum_{n=0}^{N-1}v_n^*v_{n+k} \tag{17}$$

are bounded; thus, the sequence $S(k) := \{S_N(k)\}_{N\in\mathbb{N}}$ has accumulation points and, by a Cantor-like diagonalization argument (for details see the appendix), there exists at least one subsequence $\{N_j\}_{j\in\mathbb{N}}$ such that the limit

$$s_k(v) := \lim_{j\to\infty}\frac{1}{N_j}\sum_{n=0}^{N_j-1}v_n^*v_{n+k}$$

exists for all $k \in \mathbb{N}$. By setting $s_{-k} := s_k^*$, one obtains a positive-definite sequence (details are again in the appendix), that is a sequence $s(v) = \{s_k(v)\}_{k\in\mathbb{Z}}$ such that

$$\sum_{i,j} z_i^* s_{i-j}(v) z_j \geqslant 0$$

for all sequences $\{z_i\}_{i\in\mathbb{Z}}$ such that $\sum_{i\in\mathbb{Z}}|z_i|^2 < \infty$. Then, by Bochner's theorem

$$s_k(v) = \int_0^1 \mathrm{d}\mu_v(x)\,\mathrm{e}^{2\pi\mathrm{i}kx},$$

where $\mathrm{d}\mu_v(x)$ is a positive (*correlation*) measure on [0, 1) such that

$$\int_0^1\mathrm{d}\mu_v(x) = \lim_{j\to\infty}\frac{1}{N_j}\sum_{n=0}^{N_j-1}|v_n|^2.$$

If the correlation measure of a sequence v is the Lebesgue measure, then $s_k(v) = 0$ whenever $k \neq 0$, and the sequence v is said to be *uniformly distributed*. Therefore, it makes sense to introduce the

Definition 1 (Spectrum of a sequence)**.** *[10, 11] Given a sequence $v := \{v_n\}_{n\in\mathbb{N}}$ with values in a compact subspace of $\mathbb{C}$, its Fourier–Bohr spectrum is given by*

$$\mathrm{Sp}(v) := \left\{\lambda \in [0,1) : \limsup_{N\to\infty} \frac{1}{N}\left|\sum_{n=0}^{N-1} v_n\, \mathrm{e}^{-2\pi\, \mathrm{i}n\lambda}\right| \neq 0\right\}. \tag{18}$$

In other words, the spectrum of a sequence v is the subset of values $\lambda \in [0, 2\pi)$ such that the sequences $v(\lambda) = \{v_n \exp(-2\pi \mathrm{i} n\lambda)\}_{n\in\mathbb{N}}$ are not uniformly distributed. Equivalently, $\lambda \notin \mathrm{Sp}(v)$ if and only if

$$\lim_{j\to\infty} \frac{1}{N_j} \sum_{n=0}^{N_j-1} v_n\, \mathrm{e}^{-2\pi \mathrm{i}\lambda n} = 0 \tag{19}$$

for all converging subsequences of partial sums.

By means of the spectral properties of sequences, we can derive sufficient conditions that force the invariant state ω to be tracial.

Lemma 3. *Let $v(I) := \{\mathrm{e}^{2\pi \mathrm{i} u_n(I)}\}_{n\geqslant 0}$; then, the dynamical system $(\mathcal{A}, \alpha)$ has the tracial state as its only invariant state if for each I, $\mathrm{Sp}(v(I))$ is either $\emptyset$ or $\{0\}$ with*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} v_n(I) = \sum_{j=0}^{d-1} p_j(I)\, \mathrm{e}^{\frac{2\pi \mathrm{i}}{d} j} (=: \nu(v(I))) \tag{$*$}$$

for some $p_j(I) \geqslant 0$, $j \in D$, $p_0 \neq 1$, $\sum_{j=0}^{d-1} p_j(I) = 1$.

Proof. If $\mathrm{Sp}(v(I)) = \emptyset$, then, using (19) together with the dominated convergence theorem, the right-hand side of (16) vanishes and the result follows from lemmas 2 and 1.

If for some I $\mathrm{Sp}(v(I)) = \{0\}$ and relation (*) holds for such I, then, (19) and dominated convergence applied to (16) together with lemma 2 yield

$$\langle\Omega|\pi_\omega(W_I^*) P_0^\omega \pi_\omega(W_I)|\Omega\rangle = \nu(v(I))\langle\Omega|\pi_\omega(W_I) P_0^\omega \pi_\omega(W_I^*)|\Omega\rangle. \tag{20}$$

Since $|\nu(v(I))| < 1$, by exchanging W_I and W_I^*, one gets

$$\langle\Omega|\pi_\omega(W_I^*) P_0^\omega \pi_\omega(W_I)|\Omega\rangle < \langle\Omega|\pi_\omega(W_I) P_0^\omega \pi_\omega(W_I^*)|\Omega\rangle < \langle\Omega|\pi_\omega(W_I^*) P_0^\omega \pi_\omega(W_I)|\Omega\rangle.$$

Thus, $\langle\Omega|\pi_\omega(W_I^*) P_0^\omega \pi_\omega(W_I)|\Omega\rangle = 0$, and the result follows from lemma 1. □

For some given I, for instance a singleton $I = \{1\}$, there surely exist sequences of matrices $\{A_n\}_{n\in\mathbb{N}}$, with entries from $\{0, 1, \ldots, d-1\}$, such that $\mathrm{Sp}(v(I)) = \emptyset$, or $\mathrm{Sp}(v(I)) = \{0\}$.

However, in order to use the previous lemma, we have to make sure that there exist sequences $\{A_n\}_{n\in\mathbb{N}}$ such that conditions (1) or (2) in the previous lemma are fulfilled for all I.

In the following, we shall consider the four entries $a_{ij}(n)$ of the matrices A_n as random processes with values from $\{0, 1, \ldots, d-1\}$. Then, we shall focus upon the space $\mathcal{X}$ of sequences $\vec{x} = \{\vec{x}_n\}_{n\in\mathbb{N}}$, where $\vec{x}_n = (a_{11}(n), a_{12}(n), a_{21}(n), a_{22}(n))$ are four-valued vectors with the entries of the matrices A_n as components. If we want in addition to meet the requirements in (2.2), e.g. that $\mathrm{Det}(A_{n,n}) = 1 \quad \forall n$, we can restrict to two entries $(a_{11}(n), a_{12}(n))$, but still keep enough randomness.

We equip $\mathcal{X}$ with the shift automorphism $(\sigma(\vec{x}))_n = \vec{x}_{n+1}$ and with a σ-invariant measure μ (defined on the σ algebra of cylinders). Concretely, if f is a measurable function on $\mathcal{X}$, then its mean value with respect to μ is given by

$$\mu(f) = \int_{\mathcal{X}} \mathrm{d}\mu(\vec{x})\, f(\vec{x})\ ;$$

furthermore, $\mu(f \circ \sigma) = \mu(f)$.

Observe that the quantities $\mathrm{e}^{2\pi\,\mathrm{i}\, u_n(I)}$ in (6) can be regarded as measurable functions on $\mathcal{X}$; more precisely, let P_j project out of the sequence $\vec{x}$ the jth component ($P_j\vec{x} = \vec{x}_j$). Then, consider the expression of $u_n(I)$ given by (6); it turns out that one can write

$$\mathrm{e}^{2\pi\,\mathrm{i}\, u_n(I)} = \prod_{a=1}^{n_I}\prod_{b=1}^{n_I} G_{\vec{k}_a\vec{k}_b} \circ \sigma^{b-a} \circ P_n(\vec{x}), \qquad \text{where}$$

$$G_{\vec{k}_a\vec{k}_b} \circ \sigma^{b-a} \circ P_n(\vec{x}) = \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}\sigma(\vec{k}_a, A_{b-a+n}\vec{k}_b)}. \tag{21}$$

Namely, the entries of the matrices which fix the algebraic relations (6) are given by a definite realization of the stochastic process, or by a definite trajectory of the shift-dynamical system $(\mathcal{X}, \mu, \sigma)$. We shall assume such a dynamical system to be *weakly mixing* [12] (hence *ergodic*, but less than *mixing*); this means that if f and g are two essentially bounded functions on $\mathcal{X}$ with respect to μ, then

$$\lim_{N\to+\infty} \frac{1}{N} \sum_{k=0}^{N-1} |\mu(f \cdot (g \circ \sigma^k)) - \mu(f)\mu(g)| = 0. \tag{22}$$

This condition is equivalent to the following one [12]: there exists subset $J_{fg} \subset \mathbb{N}$ such that

$$\lim_{N\to\infty} \frac{\#\Big(J_{fg} \cap \Big[0, 1, 2, \ldots, N-1\Big]\Big)}{N} = 0,$$

where # denotes the cardinality, for which

$$\lim_{n\to+\infty, n\notin J_{fg}} \mu(f \cdot (g \circ \sigma^n)) = \mu(f)\mu(g). \tag{23}$$

Definition 2 (Typical sequences)**.** *A sequence $\vec{x} \in \mathcal{X}$ will be called typical with respect to the stochastic process $(\mathcal{X}, \mu, \sigma)$ if it is self-averaging, namely if*

$$\lim_{N\to+\infty} \frac{1}{N} \sum_{k=0}^{N-1} f \circ \sigma^k(\vec{x}) = \mu(f) \tag{24}$$

for all μ-essentially bounded functions on $\mathcal{X}$ and if (23) holds with respect to $\vec{x}$, namely if

$$\lim_{n\notin J_{fg}} \lim_{N\to+\infty} \frac{1}{N} \sum_{k=0}^{N-1} f \circ \sigma^k(\vec{x}) g \circ \sigma^{n+k}(\vec{x}) = \mu(f)\mu(g) \tag{25}$$

for all μ-essentially bounded functions f, g on $\mathcal{X}$.

We now show that weak-mixing and typicality suffice to guarantee the uniqueness of the tracial state as a shift-invariant state. Fix a typical sequence $\vec{x}$ and, using the functions (21), construct the averaged sum

$$e_\lambda(\vec{x}) := \limsup_{N\to+\infty} \frac{1}{N} \sum_{k=0}^{N-1} \mathrm{e}^{-2\pi\,\mathrm{i}\lambda k} v_k, \tag{26}$$

defined by the lim sup of the real and imaginary parts of the partial sums. Because the modulus of each summand is bounded by 1, such a function exists: actually its modulus appears in (18). Note that it is not an ergodic average; however, it is such that

$$e_\lambda \circ \sigma^\ell(\vec{x}) = \limsup_{N\to+\infty} \frac{1}{N} \sum_{k=0}^{N-1} \mathrm{e}^{-2\pi\,\mathrm{i}\lambda k} v_{k+\ell} = \mathrm{e}^{2\pi\,\mathrm{i}\lambda\ell}\, e_\lambda(\vec{x}). \tag{27}$$

Proposition 1. *Let* $\{A_n\}_{n\in\mathbb{N}}$ *be a sequence of matrices provided by a typical sequence* $\vec{x}$ *of a weak-mixing stochastic process as explained and* $u(I) = \{u_n(I)\}_{n\in\mathbb{N}}$ *be the sequences defined in lemma 2; then, for all I, the spectrum of* $u(I)$ *is* $\emptyset$ *or* $\{0\}$.

Proof. Set $f(\vec{x}) = e^*_\lambda(\vec{x})$ and $g(\vec{x}) = e_\lambda(\vec{x})$ in (24) and (25); because of (27),

$$\mu(e_\lambda) = \lim_{N\to+\infty} \frac{1}{N} \sum_{k=0}^{N-1} \mathrm{e}^{-2\pi\mathrm{i}\lambda k} e_\lambda(\vec{x}) = \delta_{0\lambda} \lim_{N\to+\infty} \frac{1}{N} \sum_{k=0}^{N-1} v_k,$$

while the autocorrelation functions yield

$$\mu(e^*_\lambda e_\lambda \circ \sigma^n) = \lim_{N\to+\infty} \frac{1}{N} \sum_{k=0}^{N-1} e^*_\lambda \circ \sigma^k(\vec{x}) e_\lambda \circ \sigma^{n+k}(\vec{x}) = \mathrm{e}^{-2\pi\mathrm{i}\lambda n} |e_\lambda(\vec{x})|^2.$$

Then, (25) can hold only if $\lambda = 0$ or $e_\lambda(\vec{x}) = 0$ for $\lambda \neq 0$. In the latter case, no $\lambda \neq 0$ belongs to the spectrum of the sequence $u(I)$; on the other hand, if $\lambda = 0$, then $e_0(\vec{x}) = \mu(e_0) = \lim_{N\to+\infty} \frac{1}{N} \sum_{k=0}^{N-1} v_k$ exists as a limit and, if $e_0(\vec{x}) \neq 0$, $\lambda = 0 \in \mathrm{Sp}(u(I))$. □

Corollary 1. *Given a sequence of* 2×2 *matrices* $\{A_n\}_{n\in\mathbb{N}}$ *provided by a typical sequence in the sense of definition 2, then only the tracial state on the twisted Weyl algebra* $\mathcal{A}$ *is* α*-invariant.*

Remarks 3

(1) Given a shift-dynamical system as $(\mathcal{X}, \mu, \sigma)$, in the Koopman–von Neumann approach the shift dynamics on $\mathcal{X}$ is implemented by a unitary operator U on the Koopman Hilbert space $L^2(\mathcal{X}, \mu)$. Then, the weak-mixing condition (23) is equivalent [12] to the spectrum of U being absolutely continuous with respect to the Lebesgue measure on $[0, 2\pi)$, 1 being the only eigenvalue and the functions constant μ-almost everywhere on $\mathcal{X}$ the only eigenfunctions.

(2) Note that the function e_λ in (26) is an eigenstate of the Koopman-unitary operator U; the proof of proposition 1 is thus nothing else but the proof that weak-mixing implies that, for $\lambda \neq 0$, e_λ vanishes μ-almost everywhere. The condition of typicality selects those $\vec{x} \in \mathcal{X}$ where this is exactly the case.

(3) The weak-mixing condition appears to be the least degree of randomness that we have to ask from the underlying dynamical system $(\mathcal{X}, \mu, \sigma)$ in order to conclude as in the previous corollary. Weak mixing is implied by mixing, that is asymptotic decorrelation of two-point functions and not only on the complement of a zero-density subset,

$$\lim_{n\to+\infty} \mu(f \cdot (g \circ \sigma^n)) = \mu(f)\mu(g) \qquad \forall\ f, \quad g \in L^\infty(\mathcal{X}, \mu),$$

and implies ergodicty, that is decorrelation of two-point functions on the average,

$$\lim_{N\to+\infty} \frac{1}{N} \sum_{k=0}^{N-1} \mu(f \cdot (g \circ \sigma^k)) = \mu(f)\mu(g) \qquad \forall\ f, \quad g \in L^\infty(\mathcal{X}, \mu).$$

The easiest concrete example of dynamical system that fits into our framework, consider as the product of four identically distributed independent Bernoulli processes whose decorrelation properties are much stronger than weak mixing [12].

(4) The arguments developed in this paper can of course be applied to the Price–Powers shift. This shows that the weak-mixing condition is sufficient, but, at least in the context of example 1, not necessary. Indeed, also bitstreams not fulfilling condition (23) may have the tracial state as the only shift-invariant state. More concretely, this occurs if, for any I, there are infinitely many n_i such that $W_I\alpha^{n_i}(W_I) = -\alpha^{n_i}(W_I)W_I$ [5]. This property seemingly requires much less than typicality with respect to a weak-mixing stochastic process.

4. Conclusions

We have constructed discrete Weyl-like algebras and automorphisms on them such that they permit only one invariant state, namely the tracial state. The main tool used in the construction is the request that the algebraic relations among shifted Weyl-like operators be sufficiently irregular, and it has been implemented by means of choices of commutation relations based on typical realizations of weak-mixing stochastic processes.

Indeed, the failure of the approach reported in section 2.2 to construct invariant states does not refer to specific properties of the automorphism, rather it shows that what matters is long-range non-commutativity. If we can represent the operators as in (9) we can embody this non-commutativity in $[[\cdots[\alpha^n(A), B_0], B_1], \cdots B_n] \neq 0$ for an appropriate sequence of B_k, where the B_k is localized at the lattice point k. Therefore, the operator $\alpha^n(A)$ is not only spread as it happens for quasi-free evolution but gets delocalized also in a multiplicative sense. Nevertheless, it has still finite velocity in the sense that, at every step, an operator in the local algebra $\bigotimes_{n=0}^{N}(M_{d\times d})_n$ is mapped into an operator located in the algebra $\bigotimes_{n=0}^{N+1}(M_{d\times d})_n$.

We expect that the occurrence of non-trivial multi-commutators that do not vanish should be typical of interacting quantum systems. Of course, in general, the dynamics is such that one deals with continuous automorphism groups and with multi-commutators by far more complicated. However, the preceding analysis indicates that in the present abstract model multi-commutators are responsible for the non-existence of invariant states. This gives a hint that also in more general situations multi-commutators should play an important role in the search for invariant states.

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Appendix

The Cantor-like argument needed, following equation (17)

Define S_N as a function on the integers such that

$$S_N(k) := \frac{1}{N}\sum_{n=0}^{N-1} v_n^* v_{n+k}$$

as in (17), where $|S_N(k)| \leqslant K$ by assumption. Now $S_N(1)$ need not converge, but one can extract a subset $\{N(j,1), j = 1,2,3\ldots\} \in \mathbb{N}$ such that $S_{N(j,1)}(1)$ converges, as $j \to \infty$. Then we proceed inductively, define for each k a smaller subset $\{N(j,k), j = 1,2,3\ldots\} \subset \{N(j,k-1)\}$, such that $S_{N(j,k)}(k)$ converges as $j \to \infty$, with k fixed.

Now, and here is the Cantor diagonalization trick, one considers the set $\{N_j := N(j, j)\}$. The sequences $S_{N(j,j)}(k)$ converge, as $j \to \infty$, for each k.

Positive-definiteness

Consider a set $\{z_i\}$, $|i| \leqslant M$, extend the set $\{v_n\}$, defining $v_n := 0$ for $n < 0$, and transform

$$\sum_{i,j} z_i^* s_{i-j} z_j = \lim_{N\to\infty} \frac{1}{N} \sum_{i,j} \sum_{m=i}^{N+1+i} v_{m-i}^* z_i^* v_{m-j} z_j.$$

Considering the bounds $|v_n| < V$, $|z_i| < Z$, one gets

$$\sum_{i,j} z_i^* s_{i-j} z_j = \lim_{N\to\infty} \frac{1}{N} \left[\sum_{m\in\mathbb{Z}} \left(\sum_i v_{m-i} z_i \right)^* \left(\sum_j v_{m-j} z_j \right) + \mathrm{O}(M^2 \cdot V^2 \cdot Z^2) \right].$$

The error term vanishes in the limit $N \to \infty$ (to be taken over the subset of N where limits of the S_N exist). Then one may consider approaching ℓ^2 sequences of z_i by finite sequences.

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